

SELF INJECTIVE PROPERTY IN AMALGAMATED ALGEBRA ALONG AN IDEAL

NAJIB MAHDOU AND MOUTU ABDU SALAM MOUTUI

ABSTRACT. Let $f : A \rightarrow B$ be a ring homomorphism and let J be an ideal of B . In this paper, we investigate the transfer of self-injective property to the amalgamation of A with B along J with respect to f (denoted by $A \bowtie^f J$), introduced and studied by D'Anna, Finocchiaro and Fontana in 2009. We give also a characterization of $A \bowtie^f J$ to be quasi-Frobenius.

1. INTRODUCTION

All rings considered in this paper are assumed to be commutative, and have identity element and all modules are unitary.

Let A and B be two rings with unity, let J be an ideal of B and let $f : A \rightarrow B$ be a ring homomorphism. In this setting, we can consider the following subring of $A \times B$:

$$A \bowtie^f J := \{(a, f(a) + j) \mid a \in A, j \in J\}$$

called *the amalgamation of A and B along J with respect to f* (introduced and studied by D'Anna, Finocchiaro, and Fontana in [6, 7]). This construction is a generalization of *the amalgamated duplication of a ring along an ideal* (introduced and studied by D'Anna and Fontana in [8, 9, 10]). Moreover, other classical constructions (such as the $A + XB[X]$, $A + XB[[X]]$, and the $D + M$ constructions) can be studied as particular cases of the amalgamation ([6, Examples 2.5 and 2.6]) and other classical constructions, such as the Nagata's idealization (cf. [13, page 2]), and the CPI extensions are strictly related to it ([6, Example 2.7 and Remark 2.8]). On the other hand, the amalgamation is related to a construction proposed by Anderson in [1] and motivated by a classical construction due to Dorroh [11], concerning the embedding of a ring without identity in a ring with identity. In [6], the authors studied the basic properties of this construction (e.g., characterizations for $A \bowtie^f J$ to be a Noetherian ring, an integral domain, a reduced ring) and they characterized those distinguished pullbacks that can be expressed

2000 *Mathematics Subject Classification.* 16E05, 16E10, 16E30, 16E65.

Key words and phrases. Amalgamated algebra along an ideal, self injective, quasi-Frobenius.

as an amalgamation. Moreover, in [7], they pursued the investigation on the structure of the rings of the form $A \bowtie^f J$, with particular attention to the prime spectrum, to the chain properties and to the Krull dimension.

Self-injective rings (i.e., rings that are injective modules over themselves) play an important role in ring theory since they have connections with several kinds of rings; e.g., quasi-Frobenius rings, semiprimary rings, and Kasch rings (see [12]). In [5], The authors characterize an amalgamated duplication of a ring R along an ideal I , denoted by $R \bowtie I$ to be self-injective.

In this paper, we investigate the transfer of self-injective and quasi-Frobenius properties to amalgamation $A \bowtie^f J$ and so we generalize [5].

2. MAIN RESULTS

We first give some results of amalgamated algebra along an ideal. Recall that the modulation of A over $A \bowtie^f J$ is given via the ring map $g : A \bowtie^f J \rightarrow A$; $(a, f(a) + j) \mapsto a$ for all $a \in A$, $j \in J$. Precisely, $(a', f(a') + j).a := a'a$ for each $a, a' \in A$ and $j \in J$.

Proposition 2.1. *Let (A, B) be a pair of rings, $f : A \rightarrow B$ be an injective ring homomorphism and J be an ideal of B . Assume that $J \subseteq f(A)$. Then following isomorphism of A -modules hold:*

$$\text{Hom}_{A \bowtie^f J}(A, A \bowtie^f J) \cong J \oplus \text{Ann}_B(J).$$

Proof. Consider $\phi \in \text{Hom}_{A \bowtie^f J}(A, A \bowtie^f J)$ and set $\phi(1) = (a, f(a) + x)$ with $a \in A$ and $x \in J$. So, for each $j \in J$, $(0, 0) = \phi(0) = \phi((0, j).1) = (0, j)\phi(1) = (0, j)(a, f(a) + x) = ((0, j)f(a) + x)$. Hence, $f(a) + x \in \text{Ann}_B(J)$. Consequently, by the previous considerations, we have the following maps:

$\psi : \text{Hom}_{A \bowtie^f J}(A, A \bowtie^f J) \longrightarrow J \oplus \text{Ann}_B(J)$ and
 $g \longmapsto (x, f(a) + x)$ where $g(1) = (a, f(a) + x)$.

One can easily check that ψ is an injective homomorphism of A -modules since f is injective. It remains to show that ψ is surjective. Let $(x, j) \in J \oplus \text{Ann}_B(J)$. Since $J \subseteq f(A)$, there exist $x', j' \in f^{-1}(J)$ such that $f(j') = j$ and $f(x') = x$. Consider the $A \bowtie^f J$ -morphism defined by $g \in \text{Hom}_{A \bowtie^f J}(A, A \bowtie^f J)$ by setting, $g(1) = (x' - j', x)$. Explicitly, $g(a) = g((a, f(a) + j).1) = (a, f(a) + j)(x' - j', x) = (a(x' - j'), f(a)x)$. And so, $\psi(g) = (x' - j', x)$. Thus, ψ is an isomorphism of A -modules. \square

Let $f : A \rightarrow B$ be a ring homomorphism and J be an ideal of B . Consider the canonical (multiplication) B -map $\pi : B \rightarrow \text{Hom}_B(J, J)$ (defined by setting $\pi(b)(j) = bj$ for each $b \in B$ and $j \in J$). It is clear that $\ker(\pi) = \text{ann}_B(J)$.

Proposition 2.2. *Let (A, B) be a pair of rings, $f : A \rightarrow B$ be an injective ring homomorphism and J be an ideal of B such that $J \subseteq f(A)$ and :*

(1) *the short exact sequence of B -modules:*

(*) $0 \rightarrow \text{ann}_B(J) \hookrightarrow B \xrightarrow{\pi} \text{Hom}_B(J, J) \rightarrow 0$ *is exact and splits and,*

(2) $\text{Hom}_B(J, \text{ann}_B(J)) = 0$.

Then, $\text{Hom}_A(A \bowtie^f J, J \oplus \text{ann}_B(J))$ is isomorphic to $A \bowtie^f J$ as $A \bowtie^f J$ -module.

Proof. Since the short sequence (*) is exact and splits, there exists an A -homomorphism $\pi^{-1} : \text{Hom}_B(J, J) \rightarrow B$ such that $\pi \circ \pi^{-1}$ is the identity on $\text{Hom}_B(J, J)$. Consider the $A \bowtie^f J$ -homomorphism

$$\psi : A \bowtie^f J \rightarrow \text{Hom}_A(A \bowtie^f J, J \oplus \text{ann}_B(J))$$

defined by $\psi((1, 1)) := \phi_{(1,1)}$ where $\phi_{(1,1)}((y, f(y)+j)) = (j, (1-\pi^{-1}(\pi(1)))f(y))$ for all $y \in A$. It is easy to see that $1 - \pi^{-1}(\pi(1)) \in \text{ann}_B(J)$. Explicitly, for all $a \in A$ and $i \in J$, $\psi((a, f(a) + i)) := \phi_{(a, f(a)+i)}$ where

$$\begin{aligned} \phi_{(a, f(a)+i)}(y, f(y) + j) &= \psi((a, f(a) + i)((y, f(y) + j))) \\ &= (a, f(a) + i) \cdot \psi((1, 1))((y, f(y) + j)) \\ &= (a, f(a) + i) \cdot \phi_{(1,1)}((y, f(y) + j)) \\ &= \phi_{(1,1)}((a, f(a) + i)(y, f(y) + j)) \\ &= \phi_{(1,1)}((ay, f(a)f(y) + f(a)j + i(f(y) + j))) \\ &= (f(a)j + i(f(y) + j), (1 - \pi^{-1}(\pi(1)))f(a)f(y)) \end{aligned}$$

Recall that the natural structure of $A \bowtie^f J$ -module on $\text{Hom}_A(A \bowtie^f J, J \oplus \text{ann}_B(J))$, is defined by the scalar multiplication by $(a, f(a) + i)\phi((y, f(y) + j)) = \phi((a, f(a) + i)(y, f(y) + j))$. Consider $(a, f(a) + i) \in A \bowtie^f J$ such that $\phi_{(a, f(a)+i)} = 0$. So, $(0, 0) = \phi_{(a, f(a)+i)}(1, 1) = (i, f(a) - \pi^{-1}(\pi(f(a))))$. Therefore, $i = 0$ and $f(a) = \pi^{-1}(\pi(f(a)))$. Moreover, $(0, 0) = \phi_{(a, f(a)+i)}((0, j)) = (f(a)j + ij, 0) = (f(a)j, 0)$ for all $j \in J$. Consequently, $f(a) \in \text{ann}_B(J)$. And so $\pi(f(a)) = 0$ and $f(a) = \pi^{-1}(\pi(f(a))) = 0$. Using the fact f is injective, we obtain $a = 0$. Hence, ψ is injective.

Now, we prove that ψ is surjective. Let $\phi \in \text{Hom}_A(A \bowtie^f J, J \oplus \text{ann}_B(J))$. For all j' . The set $\phi((-j', 0)) = (\sigma_1(f(j')), \sigma_2(f(j')))$. It is clear that $\sigma_1 \in \text{Hom}(J, J)$ and $\sigma_2 \in \text{Hom}_B(J, \text{ann}_B(J))$. And so $\sigma_2 = 0$. Moreover, set $k := \pi^{-1}(\sigma_1)$. For all $j' \in f^{-1}(J)$, $\phi((-j', 0)) = (kj, 0)$ with $f(j') = j$. Also, set $\phi((1, 1)) = (i, x)$. Finally, set $f(a) = k + x$. Thus, since $x \in \text{ann}_B(J) = \ker(\pi)$, we have $\pi^{-1}(\pi(f(a))) = \pi^{-1}(\pi(k + x)) = \pi^{-1}(\pi(k)) =$

$\pi^{-1}(\pi(\pi^{-1}(\sigma_1))) = \pi^{-1}(\sigma_1) = k$ and $f(a)j = kj$ for all $j \in J$. Consequently, for each $a \in A$ and $j \in J$, using the fact $J \subset f(A)$, there exists $j' \in f^{-1}(J)$ such that $f(j') = j$. We have :

$$\begin{aligned}
& \phi((y, f(y) + j)) = \phi((y, f(y)) + (0, j)) \\
& = \phi((y, f(y)) + \phi((j', j) + (-j', 0)) \\
& = (y, f(y))\phi((1, 1)) + (j', j)\phi((1, 1)) + \phi(-j', 0) \\
& = (y, f(y))(i, x) + (j', j)(i, x) + (kj, 0) \\
& = (f(y)i, f(y)x) + (ji, 0) + (kj, 0) \\
& = (i(f(y) + j) + kj, f(y)x) \\
& = (i(f(y) + j) + kj, f(y)(f(a) - k)) \\
& = (i(f(y) + j) + f(a)j, (f(a) - \pi^{-1}(\pi(f(a))))f(y)) \\
& = \phi_{(a, f(a)+i)}(y, f(y) + j).
\end{aligned}$$

Hence, ψ is an isomorphism of $A \bowtie^f J$ -modules, as desired. \square

Remark 2.3. In particular, the conditions of the previous proposition are satisfied when $J = eB$ where e is a non zero idempotent element of B . Indeed, for each $h \in \text{Hom}_B(eB, eB)$ and each $b \in B$, $h(eb) = h(e^2b) = h(e)eb$. Hence, the canonical (multiplication) $\pi : B \rightarrow \text{Hom}_B(eB, eB)$ is surjective. Moreover, $\text{Hom}_B(eB, eB) \cong eB$ and so it is a projective module. Thus, the sequence :

(*) $0 \rightarrow \text{ann}_B(eB) \hookrightarrow B \xrightarrow{\pi} \text{Hom}_B(eB, eB) \rightarrow 0$ is exact and splits. On the other hand, for each $g \in \text{Hom}_B(eB, \text{ann}_B(eB))$, $g(eb) = g(e^2b) = g(eb)e = 0$.

The main result of this paper is the following:

Theorem 2.4. *Let (A, B) be a pair of rings, $f : A \rightarrow B$ be an injective ring homomorphism and J be an ideal of B such that $J \subseteq f(A)$. Then $A \bowtie^f J$ is a self-injective ring if and only if B is an A -module injective and there exists an idempotent element $e \in B$ such that $J = eB$.*

Proof. Assume that $A \bowtie^f J$ is a self-injective ring. By Proposition 2.13, J and $\text{ann}_B(J)$ are injective A -modules. Consider the short exact sequence of B -modules :

$$(*) 0 \rightarrow J \hookrightarrow B \xrightarrow{p} B/J \rightarrow 0.$$

We have $\text{Ext}_A^1(B/J, J) = 0$. So, (*) splits. Therefore, $B = J \oplus p^{-1}(B/J)$. Consequently, J is a principal ideal of B . Set $1 = e + g$ with $e \in J$ and $g \in p^{-1}(B/J)$. For each $x \in J$, $x = xe + xg$ and $x - xe = xg \in J \cap p^{-1}(B/J) = (0)$.

Hence, $J = eB$. Moreover, $\text{ann}_B(J) = (1 - e)B$. Thus, $B = J \oplus \text{ann}_B(J)$ is injective as an A -module.

Conversely, assume that B is an injective A -module and there exists an idempotent element e such that $J = eB$. It is clear that $\text{ann}_B(J) = (1 - e)B$. Thus, $J \oplus \text{ann}_B(J) = B$. By Proposition 2.2 and Remark 2.3, $A \bowtie^f J$ is isomorphic (as $A \bowtie^f J$ -module) to $\text{Hom}_B(A \bowtie^f J, J \oplus \text{ann}_B(J)) = \text{Hom}_A(A \bowtie^f J, B)$. Then, since B is an injective A -module, it follows that $A \bowtie^f J$ is an injective as $A \bowtie^f J$ -module and this completes the proof of Theorem 2.4. \square

The following Corollaries are consequences of Theorem 2.4.

Corollary 2.5. *Let A be a ring, B be a local ring, $f : A \rightarrow B$ be an injective ring homomorphism and let J be a non zero proper ideal of B such that $J \subseteq f(A)$. Then $A \bowtie^f J$ is never a self-injective ring.*

Proof. Since B is a local ring, then the only idempotent elements of B are $\{0, 1\}$. Hence, using the fact J is a non zero proper ideal of B and Theorem 2.4, we obtain the desired result. \square

The following Corollary is a consequence of Theorem 2.4 and is [5, Theorem 2.4].

Corollary 2.6. *Let A be a ring and let I be a ideal of A . Then $A \bowtie I$ is a self-injective ring if and only if so is A and there exists an idempotent element $e \in A$ such that $I = eA$.*

Proof. It is easy to see that $A \bowtie I = A \bowtie^f J$ where f is the identity map of A , $B = A$, $J = I$. One can easily check that $I \subset f(A)$ and f is injective. So, by Theorem 2.4, $A \bowtie I$ is a self-injective ring if and only if $B = A$ is an A -module injective and there exists an idempotent element $e \in A$ such that $J = I = eA$ and this completes the proof. \square

Now, we give a characterization of $A \bowtie^f J$ to be quasi-Frobenius. Recall that a ring is quasi-Frobenius if and only if it is Noetherian and self-injective.

Theorem 2.7. *Let (A, B) be a pair of rings, $f : A \rightarrow B$ be an injective ring homomorphism and J be an ideal of B such that $J \subseteq f(A)$. Then $A \bowtie^f J$ is quasi-Frobenius if and only if so is A , $f(A) + J$ is Noetherian, B is an*

A -module injective and there exists an idempotent element $e \in B$ such that $J = eB$.

Before proving this Theorem, we need the following Lemmas.

Lemma 2.8. [12, Theorem 1.50, 7.55 and 7.56]

For a ring A , the following statements are equivalent :

- (1) A is quasi-Frobenius.
- (2) A is Artinian and self-injective.
- (3) Every projective A -module is injective.
- (4) Every injective A -module is projective.
- (5) A is Noetherian and $\text{Ann}_A(\text{Ann}_A(J)) = J$ for every ideal J of A , where $\text{Ann}_A(J)$ denotes the annihilator of J in A .

Lemma 2.9. Let $(A_i)_{i \in I}$ be a family of commutative rings. Then $A = \prod_{i=1}^{i=n} A_i$ is quasi-Frobenius if and only if so are A_i for all $i \in I$.

Proof. Assume that $A = \prod_{i=1}^{i=n} A_i$ is quasi-Frobenius. Let $i \in I$, using [3, Proposition 2.6], $G - \text{gldim}(A) = 0$. By [4, Theorem 3.1], it follows that $G - \text{gldim}(A_i) = 0$. Hence, A_i is quasi-Frobenius for all $i \in I$. Conversely, assume that for all $i \in I$, A_i is quasi-Frobenius. By [4, Theorem 3.1], $G - \text{gldim}(A) = G - \text{gldim}(\prod_{i=1}^{i=n} A_i) = 0$. Hence, by [3, Proposition 2.6], $A = \prod_{i=1}^{i=n} A_i$ is quasi-Frobenius, as desired. \square

Lemma 2.10. Let (A, B) be a pair of rings, $f : A \rightarrow B$ be a ring homomorphism and let J be an ideal of B . If $A \bowtie^f J$ is quasi-Frobenius, then so is A .

Proof. Suppose that $A \bowtie^f J$ is quasi-Frobenius. It is easy to see that if $J = 0$, then by [6, Proposition 5.1 (3)], $A \cong \frac{A \bowtie^f J}{\{0\} \times \{J\}}$. So, $A \cong A \bowtie^f J$ which is quasi-Frobenius. If $J = B$, then $A \bowtie^f J = A \times B$. So, by Lemma 2.9, A is quasi-Frobenius. Now, assume that J is a proper ideal of B . By Lemma 2.8, $A \bowtie^f J$ is Noetherian and $\text{Ann}_{A \bowtie^f J}(\text{Ann}_{A \bowtie^f J}(L)) = L$, for every ideal L of $A \bowtie^f J$ where $\text{Ann}_{A \bowtie^f J}(-)$ is the annihilator over $A \bowtie^f J$. By [6, Proposition 5.6], A is Noetherian.

Let K be an ideal of A and our aim is to show that $\text{Ann}_A(\text{Ann}_A(K)) = K$. Clearly, $K \subseteq \text{Ann}_A(\text{Ann}_A(K))$. Conversely, let $K \bowtie^f J := \{(k, f(k) + j) / k \in K \text{ and } j \in J\}$ be an ideal of $A \bowtie^f J$. Using the fact $A \bowtie^f J$ is quasi-Frobenius, $\text{Ann}_{A \bowtie^f J}(\text{Ann}_{A \bowtie^f J}(K \bowtie^f J)) = K \bowtie^f J$. Let $(y, f(y) + h) \in \text{Ann}_{A \bowtie^f J}(K \bowtie^f J)$. Then, $\forall k \in K$, $(y, f(y) + h)(k, f(k)) = (0, 0)$. Therefore, $y \in \text{Ann}_A(K)$ and $h \in \text{Ann}_B(J)$. Now, if $x \in \text{Ann}_A(\text{Ann}_A(K))$, then $(y, f(y) + h)(x, f(x)) = (0, 0)$

and $(x, f(x)) \in \text{Ann}_{A \bowtie^f J}(\text{Ann}_{A \bowtie^f J}(K \bowtie^f J)) = K \bowtie^f J$. Hence, it follows that $x \in K$. Thus, by Lemma 2.8, A is quasi-Frobenius, as desired. \square

Proof of Theorem 2.7. Assume that $A \bowtie^f J$ is quasi-Frobenius. By Lemma 2.10, A is quasi-Frobenius. Using Lemma 2.8, $A \bowtie^f J$ is Noetherian. So, by [6, Proposition 5.6], $f(A) + J$ is Noetherian. Since f is injective and $J \subset f(A)$, then by Theorem 2.4, B is an A -module injective and there exists an idempotent element $e \in B$ such that $J = eB$, as desired.

Conversely, assume that A is quasi-Frobenius, $f(A) + J$ is Noetherian, B is an A -module injective and there exists an idempotent element $e \in B$ such that $J = eB$. By [6, Proposition 5.6] and Theorem 2.4, it follows that $A \bowtie^f J$ is quasi-Frobenius and this completes the proof of Theorem 2.7. \square

The following Corollaries follows immediately from Theorem 2.7.

Corollary 2.11. *Let A be a ring, B be a local ring, $f : A \rightarrow B$ be an injective ring homomorphism and J be a proper ideal of B such that $J \subset f(A)$. Then $A \bowtie^f J$ is never quasi-Frobenius.*

The following Corollary is a consequence of Theorem 2.4 and is [5, Proposition 2.6].

Corollary 2.12. *Let A be a ring and I be a ideal of A . Then $A \bowtie I$ is quasi-Frobenius if and only if so is A and there exists an idempotent element $e \in A$ such that $I = eA$.*

We end this paper with a characterization for $A \bowtie^f J$ to be quasi-Frobenius in a local setting. For this, we need the following lemma of independent interest.

Lemma 2.13. *Let (A, B) be a pair of rings, $f : A \rightarrow B$ be a surjective ring homomorphism and J be an ideal of B . Assume that $\text{Ann}_B(J) = 0$. Then $J \cong \text{Hom}_{A \bowtie^f J}(A, A \bowtie^f J)$.*

Proof. By [6, Proposition 5.1 (3)], $A \cong \frac{A \bowtie^f J}{\{0\} \times \{J\}}$. So, A is a cyclic $A \bowtie^f J$ -module generated (modulo $(0, J)$). Moreover, for all $a, b \in A$ and $i \in J$, $(a, f(a) + i)b = \pi_1((a, f(a) + i)b) = ab$ where $\pi_1(A \bowtie^f J) = A$. Now, for all $j \in J$, consider the following map defined by :

$$\phi : J \rightarrow \text{Hom}_{A \bowtie^f J}(A, A \bowtie^f J)$$

$j \rightarrow \psi_j$.

where $\psi_j : A \rightarrow A \bowtie^f J$ defined by $\psi_j(a) = (aj', 0)$ (where $j = f(j')$ since $J \subset f(A)$) is an $A \bowtie^f J$ -homomorphism.

Assume that $\text{Ann}_B(J) = 0$. It is easy to see that ϕ is injective. It remains to verify that ϕ is surjective. Let $h : A \rightarrow A \bowtie^f J$ be an $A \bowtie^f J$ -homomorphism and it is determined by $h(1) = (x, y)$ where $x \in A$, $y \in f(A) + J$ and $y - f(x) \in J$. Now, for all $j \in J$, we have $(xa, yf(a)) = h(a) = h((a, f(a) + j).1) = (a, f(a) + j)(x, y) = (ax, f(a)y + jy)$. So, h is well defined if and only if $yj = 0$ for all $j \in J$. Since $\text{Ann}_B(J) = 0$, then $y = 0$. Therefore, $h = \psi_{f(x)}$. Hence, $J \cong \text{Hom}_{A \bowtie^f J}(A, A \bowtie^f J)$. \square

Proposition 2.14. *Let A be a local ring, B be a ring, $f : A \rightarrow B$ be a surjective ring homomorphism and J be a non zero ideal of B . Assume that $\text{Ann}_B(J) = 0$. Then $A \bowtie^f J$ is quasi-Frobenius if and only if so is A and $J = A$.*

Proof. Assume that $A \bowtie^f J$ is quasi-Frobenius. By Lemma 2.10, A is quasi-Frobenius. Using Lemma 2.13, $J \cong_A \text{Hom}_{A \bowtie^f J}(A, A \bowtie^f J)$. So, J is an injective A -module since $A \bowtie^f J$ is self injective (quasi-Frobenius). Then, by Lemma 2.8, J is projective since A is quasi-Frobenius (by Lemma 2.8). Since A is local, J is a regular principal ideal. Let $z \in A$ be a regular element such that $J = zA$. The following descendent chain of ideals hold : $\dots z^3A \subseteq z^2A \subseteq zA$. By Lemma 2.8, A is an artinian ring. Therefore, this chain is finite and so there is an integer n such that $z^{n+1} = z^nA$. Then, there exists a non-zero element $y \in A$ such that $z^n = z^{n+1}y$ and $z^n(1 - zy) = 0$. Finally, $zy = 1$, making $J = A$, as desired. \square

REFERENCES

1. D.D. Anderson, *Commutative rings*, in : Jim Brewer, Sarah Glaz, William Heinzer, Bruce Olberding (Eds.), *Multiplicative Ideal Theory in Commutative Algebra: A tribute to the work of Robert Gilmer*, Springer, New York, 2006, pp. 1-20.
2. D. Bennis, N. Mahdou and K. Ouarghi, *Rings over which all modules are strongly Gorenstein projective*, Rocky Mountain Journal of Mathematics, Vol. 40 (3) (2010), 749 - 759.
3. D. Bennis and N. Mahdou, *Global Gorenstein Dimensions*, Proc. Amer. Math. Soc., Vol. 138 (2), (February 2010), 461-465.
4. D. Bennis and N. Mahdou, *Global Gorenstein dimensions of polynomial rings and of direct products of rings*, Houston Journal of Mathematics 25 (4), (2009), 1019-1028.
5. M. Chhiti, N. Mahdou and M. Tamekkante, *Self injective amalgamated duplication along an ideal*, J. Algebra Appl. 12, 1350033 (2013).
6. M. D'Anna, C. A. Finacchiaro, and M. Fontana, *Amalgamated algebras along an ideal*, Comm Algebra and Applications, Walter De Gruyter (2009), 241-252.

7. M. D'Anna, C. A. Finacchiaro, and M. Fontana; *Properties of chains of prime ideals in amalgamated algebras along an ideal*, J. Pure Applied Algebra **214**(2010), 1633-1641
8. M. D'Anna; *A construction of Gorenstein rings*; J. Algebra **306**(2) (2006), 507-519.
9. M. D'Anna and M. Fontana; *The amalgamated duplication of a ring along a multiplicative-canonical ideal*, Ark. Mat. **45**(2) (2007), 241-252.
10. M. D'Anna and M. Fontana; *An amalgamated duplication of a ring along an ideal: the basic properties*, J. Algebra Appl. **6**(3) (2007), 443-459.
11. J.L. Dorroh, *Concerning adjunctions to algebras*, Bull. Amer. Math. Soc. **38** (1932), 85-88.
12. W. K. Nicholson and M. F. Yousif, *Quasi-Frobenius rings*, Cambridge University Press, 2003.
13. M. Nagata, *Local Rings*, Interscience, New York, 1962.

DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE AND TECHNOLOGY OF FEZ, BOX 2202,
UNIVERSITY S. M. BEN ABDELLAH FEZ, MOROCCO
E-mail address: mahdou@hotmail.com

DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE AND TECHNOLOGY OF FEZ, BOX 2202,
UNIVERSITY S. M. BEN ABDELLAH FEZ, MOROCCO
E-mail address: moutu_2004@yahoo.fr